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Variational principles for Lagrangian-averaged fluid dynamics

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Abstract
The Lagrangian average (LA) of the ideal fluid equations preserves their fundamental transport structure. This transport structure is responsible for the Kelvin circulation theorem of the LA flow and, hence, for its potential vorticity convection and helicity conservation.

We show that Lagrangian averaging also preserves the Euler–Poincaré variational framework that implies the exact ideal fluid equations in the Eulerian representation. This is expressed in the Lagrangian-averaged Euler–Poincaré (LAEP) theorem proved here. We illustrate the LAEP theorem by applying it to incompressible ideal fluids to derive the Lagrangian-averaged Euler equations and thereby recover the generalized Lagrangian mean motion equation. Finally, we discuss recent progress in applications of these equations as the basis for new LA closure models of fluid turbulence.

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In memory of Rupert Ford (1968–2001)

1. Introduction

In turbulence, in climate modelling and in all other multiscale fluids problems, a major challenge is ‘scale-up.’ This is the challenge of deriving models for the averaged dynamics that correctly capture the mean, or large scale flow—including the influence on it of the rapid, or small scale dynamics.

The averaging performed in facing this challenge may be done in various ways. For example, meteorology and oceanography must deal with averaging in either the Eulerian, or the Lagrangian fluid specification. The Eulerian mean and the Lagrangian mean differ not in the type of average taken. (This can be ensemble average, phase average, time average, etc.) Rather, they differ in how the averaging process is related to the fluid motion. The Eulerian mean is taken at a fixed spatial location as the fluid parcels go past, while the Lagrangian mean is taken following the fluid parcels. The Eulerian mean commutes with both the spatial gradient and the partial time derivative at fixed position; so it has the advantage of preserving the momentum-conservation form of the hydrodynamics equations. However, the Eulerian mean does not commute with the advective time derivative; so it fails to preserve important circulation properties such as conservation of potential vorticity on fluid parcels.

Vice versa, the Lagrangian mean has the advantage of preserving the fundamental *transport structure* of fluid dynamics. In particular, the Lagrangian mean commutes with the advective time derivative moving with the flow. Therefore, the Lagrangian mean preserves the Kelvin circulation property of the fluid motion equation and potential vorticity conservation on fluid parcels. However, the Lagrangian mean also has two main *disadvantages*: it is history dependent (since it must follow the fluid parcels); and it does not commute with the spatial gradient.

Determining the relation between averaged quantities obtained in these two equivalent specifications of fluid dynamics is one of the classical problems in the physics of fluids. Many attempts have been made to establish practicable relations between the Eulerian mean and the Lagrangian mean. One famous example is G I Taylor's hypothesis that turbulent fluctuations are 'frozen' into the mean flow. According to Taylor's hypothesis, a time series of turbulent quantities measured at one location may be interpreted approximately as arising from an upstream spatial distribution of fluctuations at an earlier time being swept downstream by the Eulerian mean flow velocity [1].

The *generalized Lagrangian mean* (GLM) theory of Andrews and McIntyre [2] implements essentially the converse of Taylor's hypothesis and thereby systematizes the relation of Lagrangian mean fluid equations to their Eulerian mean counterparts. GLM theory begins by introducing a slow and fast decomposition of the Lagrangian parcel trajectory in general form. In these exact (but not closed) equations, the Lagrangian mean of a fluid quantity evaluated at the *mean* fluid parcel position is related to its Eulerian mean, evaluated at the *current* fluid parcel position. The precise relation depends on the tensor transformation properties of the quantity being averaged. Thus, the GLM equations express the Lagrangian mean fluid dynamics directly in the Eulerian representation.

In this paper, we place Lagrangian averaged (LA) fluid equations such as the GLM equations into the Euler–Poincaré (EP) framework of constrained variational principles [5–7]. This demonstrates the variational reduction property of the Lagrangian mean, encapsulated in the theorem proved here:

Lagrangian-averaged Euler–Poincaré (LAEP) theorem. *Lagrangian-averaging (LA) preserves the variational structure of the Euler–Poincaré (EP) framework for fluid dynamics.*

According to the LAEP theorem, preservation of the fundamental transport structure of fluid dynamics by the Lagrangian-average (LA) extends to preserving its Euler–Poincaré (EP) variational structure [5–7]. That is, Lagrangian-averaging preserves the four equivalence relations of the EP theorem, as we show in section 3. Consequently, the GLM equations, in particular, follow from the Lagrangian-averaged EP variational principle for the exact Euler equations. This preservation of variational structure is *not* possible with the Eulerian mean. The Eulerian mean also does not preserve the transport structure of fluid mechanics, e.g. the Kelvin circulation theorem, that follows as a corollary of the EP variational structure.

Thus, the LAEP theorem proved here puts the two modelling approaches using either LA Hamilton's principles (such as Whitham's averaged Lagrangian method), or LA equations (such as the GLM method) on an equal footing. This is quite a bonus for both approaches to modelling fluid dynamics. The LAEP theorem implies, in particular, that the LA Hamilton's principle produces dynamics that is guaranteed to be verified directly by Lagrangian-averaging the original equations, and these LA equations inherit the conservation laws and balance laws that are available from the symmetries of Hamilton's principle for fluids.

1.1. Outline of the paper

We begin by briefly reviewing the GLM theory of Andrews and McIntyre [2]. We then state and prove the LAEP theorem, following the Euler–Poincaré (EP) framework established in [5–7]. We shall illustrate the LAEP theorem by applying it to incompressible ideal fluids. This will recover the familiar GLM motion equations for this case. Finally, we shall discuss recent progress toward closure of these LA equations in the development of new models of fluid turbulence that are based on Lagrangian-averaging and also include the effects of viscosity.

2. Generalized Lagrangian mean theory

In ideal fluid dynamics, the Lagrange-to-Euler map gives the current position of a fluid parcel that was initially at position $\boldsymbol{x}_0$. This map is assumed to be a diffeomorphism $g(t)$ (a smooth invertible map with a continuous inverse) parametrized by time t. Diffeomorphisms may be composed, so the Lagrange-to-Euler map may be expressed equivalently as a composition of two other diffeomorphisms, denoted as $g(t) = \Xi(t) \cdot \tilde{g}(t)$ and subject to the chain rule under differentiation.

The GLM theory [2] starts by applying an averaging process to the Lagrange-to-Euler map that holds the fluid label $\boldsymbol{x}_0$ fixed. The averaging process $\overline{(\cdot)}$ can be reasonably arbitrary, except that it must be consistent with the diffeomorphism group, so that the Lagrange-to-Euler map for the average fluid trajectory is again expressible as a diffeomorphism, denoted $\bar{g}(t)$. This is the *key premise* of GLM theory. Thus, in the decomposition $g(t) = \Xi(t) \cdot \tilde{g}(t)$, the GLM averaging process may be expressed as $\bar{g}(t) = \overline{\Xi(t) \cdot \tilde{g}(t)}$ and we may choose $\tilde{g}(t) = \bar{g}(t)$, since GLM averaging is supposed to be consistent with the diffeomorphisms. GLM theory also requires the averaging process to satisfy the *projection property*, so that $\bar{\bar{g}}(t) = \bar{\tilde{g}}(t) = \tilde{g}(t)$. Hence, a fluid parcel labelled by $\boldsymbol{x}_0$ has current position

$$\boldsymbol{x}^{\xi}(\boldsymbol{x}_0, t) \equiv g(t) \cdot \boldsymbol{x}_0 = \Xi(t) \cdot (\tilde{g}(t) \cdot \boldsymbol{x}_0) = \Xi(\boldsymbol{x}(\boldsymbol{x}_0, t), t)$$

and it has mean position $\boldsymbol{x}(\boldsymbol{x}_0, t) = \tilde{g}(t) \cdot \boldsymbol{x}_0 = \overline{\boldsymbol{x}^{\xi}}(\boldsymbol{x}_0, t)$. For example, the Lagrangian-averaging process used in the WKB representation for fluid fluctuations in Gjaja and Holm [3] satisfies these conditions and may be represented this way. See figure 1 for a schematic representation of this composition of maps.

2.1. GLM velocities and advective derivatives

The composition of maps $g(t) = \Xi(t) \cdot \tilde{g}(t)$ yields via the chain rule the following *velocity relation*

$$\dot{\boldsymbol{x}}^{\xi}(\boldsymbol{x}_0, t) = \dot{g}(t) \cdot \boldsymbol{x}_0 = \dot{\Xi}(t) \cdot \boldsymbol{x} + T\,\Xi \cdot (\dot{\tilde{g}}(t) \cdot \boldsymbol{x}_0). \tag{1}$$

By invertibility, $\boldsymbol{x}_0 = g^{-1}(t) \cdot \boldsymbol{x}^{\xi} = \tilde{g}^{-1}(t) \cdot \boldsymbol{x}$, for the fluid parcel initially at position $\boldsymbol{x}_0$. Hence, one may define each fluid parcel's velocity at its current position $\boldsymbol{u}(\boldsymbol{x}^{\xi}, t)$ in terms of a vector field evaluated at its mean position $\boldsymbol{u}^{\xi}(\boldsymbol{x}, t)$ as

$$\boldsymbol{u}(\boldsymbol{x}^{\xi}, t) = \dot{g} \cdot g^{-1}(t) \cdot \boldsymbol{x}^{\xi} = \dot{g} \cdot \tilde{g}^{-1}(t) \cdot \boldsymbol{x} \equiv \boldsymbol{u}^{\xi}(\boldsymbol{x}, t).$$

The velocity relation (1) then implies

$$\boldsymbol{u}^{\xi}(\boldsymbol{x}, t) = \frac{\partial \Xi}{\partial t}(\boldsymbol{x}, t) + \frac{\partial \Xi}{\partial \boldsymbol{x}} \cdot \bar{\boldsymbol{u}}^{L}(\boldsymbol{x}, t). \tag{2}$$

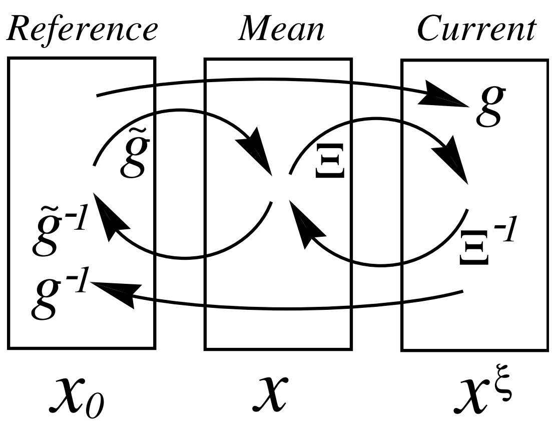

Figure 1. GLM theory factorizes the Lagrange-to-Euler map at a given time by first mapping the reference configuration to the mean position, then mapping that to the current position.

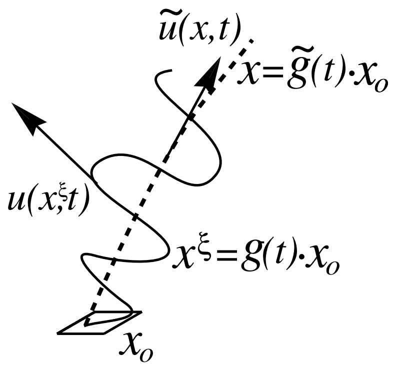

Figure 2. The GLM velocities $\boldsymbol{u}(\boldsymbol{x}^{\xi},t)$ and $\bar{\boldsymbol{u}}^L(\boldsymbol{x},t)$ are tangent to the current and mean trajectories, $\boldsymbol{x}^{\xi}$ and $\boldsymbol{x}$, respectively.

This is a standard velocity relation from GLM theory, in which the *Lagrangian mean velocity* $\bar{\boldsymbol{u}}^L$ is *defined* as

$$\bar{\boldsymbol{u}}^L(\boldsymbol{x},t) \equiv \overline{\boldsymbol{u}^{\xi}(\boldsymbol{x},t)} = \overline{\dot{g}\tilde{g}^{-1}(t)} \cdot \boldsymbol{x} = \dot{\tilde{g}}(t)\tilde{g}(t)^{-1} \cdot \boldsymbol{x}. \tag{3}$$

In the third equality one invokes the projection property of the averaging process as $\overline{\tilde{g}^{-1}(t)} = \tilde{g}(t)^{-1}$ and finds $\bar{\dot{g}} = \dot{\bar{g}} = \dot{\tilde{g}}$ from equation (1), so that $\bar{\boldsymbol{u}}^L(\boldsymbol{x},t) = \dot{\tilde{g}}(t)\tilde{g}(t)^{-1} \cdot \boldsymbol{x} \equiv \tilde{\boldsymbol{u}}(\boldsymbol{x},t)$. Thus, the Lagrangian mean velocity $\bar{\boldsymbol{u}}^L$ coincides with $\tilde{\boldsymbol{u}}$, the vector field tangent to the mean motion associated with $\tilde{g}(t)$. See figure 2 for a schematic representation of this tangency property. Hence, one may write the GLM velocity decomposition (2) in terms of the *LA material time derivative* $\mathrm{D}^L/\mathrm{D}t$ as

$$\boldsymbol{u}^{\xi}(\boldsymbol{x},t) = \left(\frac{\partial}{\partial t} + \bar{\boldsymbol{u}}^L \cdot \nabla\right) \Xi(\boldsymbol{x},t) \equiv \frac{\mathrm{D}^L}{\mathrm{D}t} \Xi(\boldsymbol{x},t). \tag{4}$$

For any other fluid quantity χ one may similarly define χ^{ξ} as the composition of functions $\chi^{\xi}(\boldsymbol{x},t) = \chi(\boldsymbol{x}^{\xi},t) = \chi(\Xi(\boldsymbol{x},t),t)$. Taking the LA material time derivative of χ^{ξ} and using the definition of $\mathrm{D}^L/\mathrm{D}t$ in equation (4) yields the *advective derivative relation*

$$\left(\frac{\partial \chi}{\partial t}\right)^{\xi} + T\chi \cdot \frac{\mathrm{D}^L}{\mathrm{D}t} \Xi(\boldsymbol{x},t) = \left(\frac{\partial \chi}{\partial t} + T\chi \cdot \boldsymbol{u}\right)^{\xi} \tag{5}$$

so $\mathrm{D}^L\chi^\xi/\mathrm{D}t = (\mathrm{D}\chi/\mathrm{D}t)^\xi$. As in equation (3) for the velocity, the *Lagrangian mean* $\bar\chi^L$ of a fluid quantity χ is defined as

$$\bar\chi^L(\boldsymbol{x},t) \equiv \overline{\chi^\xi(\boldsymbol{x},t)} = \overline{\chi(\boldsymbol{x}^\xi,t)} = \overline{\chi(\,g(t)\cdot\boldsymbol{x}_0,t)}. \tag{6}$$

Taking the Lagrangian mean of equation (5) and again using its projection property yields $\dot{\bar\chi}^L = \mathrm{D}^L\bar\chi^L/\mathrm{D}t = \overline{(\mathrm{D}\chi/\mathrm{D}t)}^L = \bar{\dot\chi}^L$. Thus, as expected, the Lagrangian mean defined in (6) commutes with the material derivative.

2.2. *Transformation factors of advected quantities*

Advective transport by $g(t)$ and $\tilde g(t)$ is defined by group action from the right

$$a(\boldsymbol{x}^\xi,t) = a_0\cdot g^{-1}(t) \qquad \text{and} \qquad \tilde a(\boldsymbol{x},t) = a_0\cdot\tilde g^{-1}(t)$$

where $a_0 = a(\boldsymbol{x}_0,0) = \tilde a(\boldsymbol{x}_0,0)$, with $a,\tilde a\in V^*$. The factorization $g(t) = \Xi\,(t)\cdot\tilde g(t)$ implies $\tilde a(\boldsymbol{x},t) = a\cdot\,\Xi\,(\boldsymbol{x},t)$. Since a and $\tilde a$ refer to the *same* initial conditions, a_0, we have

$$a_0\cdot\tilde g^{-1}(t) = \tilde a(\boldsymbol{x},t) = a\cdot\,\Xi\,(\boldsymbol{x},t) \equiv \mathcal{F}(\boldsymbol{x},t)\cdot a^\xi(\boldsymbol{x},t). \tag{7}$$

Note that the right side of this equation is potentially rapidly varying, but the left side is a mean advected quantity. Here $\mathcal{F}(\boldsymbol{x},t)$ is the *tensor transformation factor* of the advected quantity a under the change of variables $\Xi\colon \boldsymbol{x}\to\boldsymbol{x}^\xi$. For example, the density, D, transforms as

$$D^\xi\,\det(T\,\Xi)(\boldsymbol{x},t) = \tilde D(\boldsymbol{x},t) \qquad \mathcal{F}(\boldsymbol{x},t) = \det(T\,\Xi) \tag{8}$$

$$\text{and } \tilde D \text{ advects as} \qquad \partial_t\tilde D = -\mathrm{div}(\tilde D\tilde{\boldsymbol{u}}). \tag{9}$$

The transformation factors are 1, $\det(T\,\Xi)$ and $\mathsf{K}\equiv\det(T\,\Xi\,)\,T\,\Xi^{-1}$, for advected scalar, density and vector fields, respectively. In each case, the corresponding transformation factor $\mathcal{F}$ appears in a *variational relation* for an advected quantity, expressed via equation (7) as

$$\delta a^\xi = \delta\,(\mathcal{F}^{-1}\cdot\tilde a) = \mathcal{F}^{-1}\cdot\delta\tilde a + (\delta\mathcal{F}^{-1})\cdot\tilde a. \tag{10}$$

This formula will be instrumental in establishing the main result of this paper given in the next section.

3. Lagrangian-averaged Euler–Poincaré theorem (LAEP)

Let the following list of assumptions hold [5, 6].

- There is a *right* representation of Lie group G on the vector space V and G acts in the natural way on the *right* on $TG\times V^*$: $(v_g,a)h = (v_gh,ah)$ where V^* is the dual space of V.
- The function $L\colon TG\times V^*\to\mathbb{R}$ is right G-invariant.
- In particular, if $a_0\in V^*$, define the Lagrangian $L_{a_0}\colon TG\to\mathbb{R}$ by $L_{a_0}(v_g) = L(v_g,a_0)$. Then L_{a_0} is right-invariant under the lift to TG of the right action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.
- Right G-invariance of L permits one to define $\ell\colon\mathfrak{g}\times V^*\to\mathbb{R}$ by $\ell\left(v_gg^{-1},a_0g^{-1}\right) = L(v_g,a_0)$. Conversely, this relation defines for any $\ell\colon\mathfrak{g}\times V^*\to\mathbb{R}$ a right G-invariant function $L\colon TG\times V^*\to\mathbb{R}$.
- For a curve $g(t)\in G$, let $u(t)\equiv\dot g(t)g(t)^{-1}\in TG/G\simeq\mathfrak{g}$ and define the curve $a(t)$ as the unique solution of the linear differential equation with time dependent coefficients $\dot a(t) = -a(t)u(t)$ where the action of $u\in\mathfrak{g}$ on the initial condition $a(0) = a_0\in V^*$ is denoted by concatenation from the right. This solution can be written as the *advective transport relation*, $a(t) = a_0g(t)^{-1}$.
- The GLM factorization holds, $g(t) = \Xi\,(t)\cdot\tilde g(t)$, in which the average defined as $\bar g(t) = \overline{\Xi\,(t)\cdot\tilde g(t)} = \tilde g(t)$ satisfies the projection property $\bar{\bar g}(t) = \tilde g(t)$.

3.1. LAEP theorem

The following four statements are equivalent:

(i) The averaged Hamilton's principle holds

$$\delta\int_{t_1}^{t_2}\overline{L_{a_0}(g(t),\dot g(t))}\,\mathrm{d}t=0 \tag{11}$$

for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

(ii) The averaged Euler–Lagrange equations for $\bar L_{a_0}$ are satisfied on $T^*\tilde G$

$$\overline{\frac{\delta L_{a_0}}{\delta g}\cdot T\,\Xi}-\overline{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L_{a_0}}{\delta\dot g}\cdot T\,\Xi}=0 \tag{12}$$

(iii) The averaged constrained variational principle

$$\delta\int_{t_1}^{t_2}\overline{\ell(u(t),a(t))}\,\mathrm{d}t=0 \tag{13}$$

holds, using the chain-rule induced variations

$$\begin{aligned}\delta u&=(\partial_t+\mathrm{ad}_u)\eta'+(T\,\Xi\cdot(\partial_t+\mathrm{ad}_{\tilde u})\tilde\eta)\,\Xi^{-1}\\ \delta a&=-a\eta=\delta(\mathcal{F}^{-1}\cdot\tilde a)\,\Xi^{-1}\\ &=-a\eta'-(\mathcal{F}^{-1}\cdot(\tilde a\tilde\eta))\,\Xi^{-1}\end{aligned} \tag{14}$$

where Lie derivatives of advected quantities by the vector fields

$$\eta'(t)\equiv\delta\Xi\,\Xi^{-1}\qquad\tilde\eta(t)\equiv\delta\tilde g\tilde g^{-1}\qquad\text{and}\qquad\eta(t)\equiv\delta g g^{-1}=\eta'+(T\,\Xi\cdot\tilde\eta)\,\Xi^{-1} \tag{15}$$

are indicated by concatenation on the right and these three vector fields all vanish at the endpoints.

(iv) The Euler–Poincaré (EP) equation holds on $\mathfrak{g}\times V^*$

$$\left(\frac{\partial}{\partial t}+\mathrm{ad}^*_u\right)\frac{\delta\ell}{\delta u}=\frac{\delta\ell}{\delta a}\diamond a \tag{16}$$

and the Lagrangian averaged Euler–Poincaré (LAEP) equation holds on $\tilde{\mathfrak{g}}\times\tilde V^*$

$$\left(\frac{\partial}{\partial t}+\mathrm{ad}^*_{\tilde u}\right)\overline{\left(\frac{\delta\ell}{\delta u^\xi}\cdot T\,\Xi\right)}=\overline{\left(\frac{\delta\ell}{\delta a^\xi}\cdot\mathcal{F}^{-1}\right)}\diamond\tilde a\,. \tag{17}$$

Notation. In equations (16) and (17), the operations ad^* and $\diamond$ are defined by using the L_2 pairing $\langle f,g\rangle=\int fg\,\mathrm{d}^3x$. The ad^* operation is defined as (minus) the L_2 dual of the Lie algebra operation, ad, or commutator, $\mathrm{ad}_u\eta=-[u,\eta]$, for vector fields, $-\langle\mathrm{ad}^*_u\mu,\eta\rangle\equiv\langle\mu,\mathrm{ad}_u\,\eta\rangle$. The diamond operation $\diamond$ is defined as (minus) the L_2 dual of the Lie derivative, namely, $\langle b\diamond a\,,\,\eta\rangle\equiv-\langle b\,,\,\pounds_\eta a\rangle=-\langle b\,,\,a\,\eta\rangle$, where $\pounds_\eta a$ denotes the Lie derivative with respect to vector field η of the tensor a, and a and b are dual tensors under the L_2 pairing.

Proof of the LAEP theorem. The equivalence of (i) and (ii) may be shown by a direct computation. To compute the averaged Euler–Lagrange equation (12), we use the following *variational relation* obtained from the composition of maps $g(t)=\Xi\,(t)\cdot\tilde g(t)$, cf the velocity relation (1)

$$\delta g(t)=\delta\,\Xi\,(t)\cdot\tilde g(t)+T\,\Xi\,(t)\cdot\delta\tilde g(t). \tag{18}$$

Hence, after integrating by parts and using the projection property we find that

$$
\begin{aligned}
0 = \delta \int_{t_1}^{t_2} \overline{L_{a_0}(g(t), \dot{g}(t))}\, \mathrm{d}t &= \int_{t_1}^{t_2} \left(\overline{\frac{\delta L_{a_0}}{\delta g} \cdot \delta g} + \overline{\frac{\delta L_{a_0}}{\delta \dot{g}} \cdot \delta \dot{g}} \right) \mathrm{d}t \\
&= \int_{t_1}^{t_2} \overline{\left(\left(\frac{\delta L_{a_0}}{\delta g} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta L_{a_0}}{\delta \dot{g}} \right) \cdot \delta \Xi\,(t) \right)} \cdot \tilde{g}\, \mathrm{d}t \\
&\quad + \int_{t_1}^{t_2} \left(\overline{\frac{\delta L_{a_0}}{\delta g} \cdot T\,\Xi} - \overline{\frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta L_{a_0}}{\delta \dot{g}} \cdot T\,\Xi} \right) \cdot \delta \tilde{g}\, \mathrm{d}t.
\end{aligned} \tag{19}
$$

In the last equality, the first of the two integrals vanishes for any $\delta\,\Xi$, thereby ensuring that the Euler–Lagrange equations

$$
\frac{\delta L_{a_0}}{\delta g} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta L_{a_0}}{\delta \dot{g}} = 0
$$

are satisfied *before* averaging is applied. The vanishing of the second of these two integrals for arbitrary $\delta\tilde{g}$ then yields the averaged Euler–Lagrange equations (12), in which the transformation factor $T\,\Xi$ is contracted with the Euler–Lagrange equations before averaging is applied.

The equivalence of (iii) and (iv) in the LAEP theorem now follows by substituting the variations (14) defined using the chain rule into (13) and integrating by parts to obtain

$$
0 = \delta \int_{t_1}^{t_2} \overline{\ell(u, a)}\ \mathrm{d}t = \int_{t_1}^{t_2} \overline{\left\langle \frac{\delta \ell}{\delta u}, \delta u \right\rangle} + \overline{\left\langle \frac{\delta \ell}{\delta a}\ ,\ \delta a \right\rangle}\ \mathrm{d}t
$$

$$
= -\int_{t_1}^{t_2} \overline{\left\langle (\partial_t + \mathrm{ad}^*_u) \frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta a} \diamond a, \eta' \right\rangle}\ \mathrm{d}t \tag{20}
$$

$$
- \int_{t_1}^{t_2} \left\langle \left(\frac{\partial}{\partial t} + \mathrm{ad}^*_{\tilde{u}} \right) \overline{\left(\frac{\delta \ell}{\delta u^\xi} \cdot T\,\Xi \right)} - \overline{\left(\frac{\delta \ell}{\delta a^\xi} \cdot \mathcal{F}^{-1} \right)} \diamond \tilde{a}\ ,\ \tilde{\eta} \right\rangle \mathrm{d}t. \tag{21}
$$

Thus, the independent variations η' in (20) and $\tilde{\eta}$ in (21) result in the EP motion equation (16) and the LAEP motion equation (17), respectively.

Finally we show that (i) and (iii) are equivalent. First note that the G-invariance of $L\colon TG \times V^* \to \mathbb{R}$ and the definition of $a(t) = a_0 g(t)^{-1}$ imply that the integrands in (11) and (13) are equal. In fact, this holds both before and after averaging. Moreover, all variations $\delta g(t) \in TG$ of $g(t)$ with fixed endpoints induce and are induced by variations $\delta u(t) \in \mathfrak{g}$ of $u(t)$ of the form $\delta u = \partial\eta/\partial t + \mathrm{ad}_u\,\eta$ with $\eta(t) \in \mathfrak{g}$ vanishing at the endpoints. The relation between $\delta g(t)$ and $\eta(t)$ is given by $\eta(t) = \delta g(t) g(t)^{-1}$. The corresponding statements also hold for the prime- and tilde-variables in the variational relations (14) that are used in the calculation of the other equivalences. These observations show that (i) and (iii) are also equivalent, and this finishes the proof of the LAEP theorem.

Lie derivative versus ad*. The equality $\mathrm{ad}^*_u\mu = \pounds_u\mu$ holds for any one-form density μ (such as $\mu = \delta\ell/\delta u$, the variational derivative). Thus, the EP motion equation (16) and the LAEP motion equation (17) may be written equivalently using Lie derivatives as, respectively

$$
\left(\frac{\partial}{\partial t} + \pounds_u \right) \frac{\delta \ell}{\delta u} = \frac{\delta \ell}{\delta a} \diamond a \tag{22}
$$

$$
\left(\frac{\partial}{\partial t} + \pounds_{\tilde{u}} \right) \overline{\left(\frac{\delta \ell}{\delta u^\xi} \cdot T\,\Xi \right)} = \overline{\left(\frac{\delta \ell}{\delta a^\xi} \cdot \mathcal{F}^{-1} \right)} \diamond \tilde{a}\,. \tag{23}
$$

In this notation, the advection of mass by the LA motion takes the form $(\partial_t + \pounds_{\tilde{u}})\tilde{D} = 0$ and equation (23) immediately implies the following corollary of the LAEP theorem.

3.2. LA Kelvin–Noether circulation theorem

The one-form $\tilde{v} \equiv \overline{\left(\frac{\delta\ell}{\delta u^\xi} \cdot T\,\Xi\right)}/\tilde{D}$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(\tilde{u})} \tilde{v} = \oint_{c(\tilde{u})} \frac{1}{\tilde{D}}\,\overline{\left(\frac{\delta\ell}{\delta a^\xi} \cdot \mathcal{F}^{-1}\right)} \diamond \tilde{a}$$

for any closed curve $c(\tilde{u})$ *following the LA fluid motion.*

This corollary follows from the LA motion equation written as (23) and the LA mass conservation law, $(\partial_t + \pounds_{\tilde{u}})\tilde{D} = 0$. Because of the equivalence relations in the LAEP theorem, the *same* result may be obtained by applying LA to Kelvin's theorem for the exact EP motion equation (16)

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{c(u)} \frac{1}{D}\frac{\delta\ell}{\delta u} = \oint_{c(u)} \frac{1}{D}\frac{\delta\ell}{\delta a} \diamond a\,.$$

This exact Kelvin's theorem is easily derived from the form (22) of the exact motion equation, upon using exact mass conservation in the form $(\partial_t + \pounds_u)D = 0$.

3.3. Applying the LAEP theorem to incompressible fluids

The Lagrangian averaged Euler (LAE) equations for an incompressible fluid are derived from the LAEP theorem by using the reduced averaged Lagrangian

$$\bar{\ell} = \int \mathrm{d}^3x \left[\frac{1}{2}\tilde{D}\,\overline{|\boldsymbol{u}^\xi|^2} + \overline{p^\xi\,(\det T\,\Xi - \tilde{D})}\right] \tag{24}$$

which was obtained as the LA of Hamilton's principle for Euler's equations given, for example, in [5]. The pressure constraint implies that the mean advected density is related to the mean fluid trajectory by $\tilde{D} = \overline{\det T\,\Xi}$. Thus, in general, the LAE fluid velocity has a nonzero divergence [2], since (9) for $\tilde{D}$ implies

$$\operatorname{div}\tilde{\boldsymbol{u}} = -\frac{1}{\tilde{D}}\left(\frac{\partial}{\partial t} + \bar{\boldsymbol{u}}^L\cdot\nabla\right)\tilde{D} \neq 0.$$

This is to be expected, since LA does not commute with the spatial gradient. In principle, one may restrict $g(t)$ and both its factors $\Xi(t)$ and $\tilde{g}(t)$ to the space of volume-preserving diffeomorphisms, for which $\det T\,\Xi \equiv 1$. However, some LA processes may not respect this restriction and, in general, $\operatorname{div}\tilde{\boldsymbol{u}} \neq 0$. In [8, 9] $\operatorname{div}\tilde{\boldsymbol{u}} = 0$ is accomplished for the type of averaging defined there, which differs from LA as we discuss it here.

The GLM motion equation. For $\bar{\ell}$ in (24) the LAEP equation (17) gives the LAE equations

$$\frac{\partial}{\partial t}\tilde{v}_i + \tilde{u}^j\frac{\partial}{\partial x^j}\tilde{v}_i + \tilde{v}_j\frac{\partial}{\partial x^i}\tilde{u}^j + \frac{\partial}{\partial x^i}\tilde{\pi} = 0 \qquad \text{and} \qquad \partial_t\tilde{D} = -\operatorname{div}(\tilde{D}\tilde{\boldsymbol{u}}) \tag{25}$$

with mean fluid quantities $\tilde{v}_i$ and $\tilde{\pi}$ defined as

$$\tilde{v}_i = \frac{1}{\tilde{D}}\frac{\delta\bar{\ell}}{\delta\tilde{u}^i} = \overline{u_j^\xi(T\,\Xi)_i^j} \qquad \tilde{\pi} = -\frac{\delta\bar{\ell}}{\delta\tilde{D}} = -\frac{1}{2}\overline{|\boldsymbol{u}^\xi|^2} + \bar{p}^L.$$

When $T\,\Xi = Id + \nabla\xi$, for a vector field $\xi = \boldsymbol{x}^\xi - \boldsymbol{x}$ as in [2], one finds

$$\tilde{\boldsymbol{v}} = \overline{\boldsymbol{u}^\xi} + \overline{\frac{\mathrm{D}^L}{\mathrm{D}t}\xi_j\nabla\xi^j} \equiv \bar{\boldsymbol{u}}^L - \bar{\boldsymbol{p}}. \tag{26}$$

Hence, the LAEP equations (25) for the reduced averaged Lagrangian (24) recover the GLM motion equation, with the GLM pseudomomentum $\bar{\boldsymbol{p}} = -\overline{\frac{\mathrm{D}^L}{\mathrm{D}t}\xi_j \nabla \xi^j}$ of [2]. See, e.g., [2, 3, 4, 11, 12] for discussions of the role that pseudomomentum plays in GLM theory.

Momentum balance. *Following the EP theory of Holm, Marsden and Ratiu [5, 6] leads to the momentum balance relation for the LAE equations (25)*

$$\frac{\partial}{\partial t}(\tilde{D}\tilde{v}_i) + \frac{\partial}{\partial x^j}\left(\tilde{D}\tilde{v}_i\tilde{u}^j + \bar{p}^L\delta_i^j\right) = \frac{\tilde{D}}{2}\frac{\partial\overline{|\boldsymbol{u}^\xi|^2}}{\partial x^i}\bigg|_{\exp} \tag{27}$$

where subscript exp refers to the explicit spatial dependence that yields a mean force arising from the Ξ-terms *in* $\overline{|\boldsymbol{u}^\xi|^2} = \overline{|\mathrm{D}^L\,\Xi\,/\mathrm{D}t\,|^2}$ *that appear in equation (2).*

Proof. This LA momentum balance relation follows when Noether's theorem is applied to the reduced averaged Lagrangian (24) for the LAE equations. See [5, 6] for discussions of Noether's theorem in the EP context for continuum mechanics. □

4. Recent progress towards closure and applications of LA in turbulence modelling

Of course, the LAE equations (25) are not yet closed. As indicated in their momentum balance relation (27), they depend on the unspecified Lagrangian statistical properties appearing as the Ξ-terms in the definitions of $\tilde{\boldsymbol{v}}$ and $\tilde{\pi}$. Until these properties are modelled or prescribed, the LAE equations are incomplete.

Progress in formulating and analysing a closed system of fluid equations based on the nonlinear transport properties of the LAE equations has recently been made in the Euler–Poincaré context. The closed model LAE equations were first obtained in Holm, Marsden and Ratiu [5, 6] by using Taylor's hypothesis as a closure step. A self-consistent variant of the LAE closure including the effects of buoyancy was also introduced in Gjaja and Holm [3] in the Lagrangian fluid specification by using a WKB approximation for the fluctuating vector field $\xi = \boldsymbol{x}^\xi - \boldsymbol{x}$. See Holm [11, 12] for further discussion of that approach, which uses asymptotic expansions of Hamilton's principle for GLM to order $O(|\xi|^2)$ in combination with Taylor's hypothesis in developing the closure equations.

This type of closure method has recently been developed to the point of application as the basis of a turbulence model (after properly including viscous dissipation) in Chen *et al* [13, 14, 15, 16]. This LANS-α model—the Lagrangian-averaged Navier–Stokes-α equations—was compared to large eddy simulation (LES) methods in Domaradzki and Holm [17], Mohseni *et al* [18], Holm and Kerr [19] and Geurts and Holm [20]. See Shkoller [21], Holm [22], Foias *et al* [23, 24] Marsden *et al* [8] and Marsden and Shkoller [9], [10] for additional mathematical studies and discussions of the LANS-α equations.

Of course, the LAEP approach is also versatile enough to derive LA equations for *compressible* fluid motion. In fact, this was already shown in the original GLM theory [2]. For brevity now, we only remark that the LAEP approach preserves helicity conservation for barotropic compressible flows. It also preserves magnetic helicity and cross-helicity conservation when applied to magnetohydrodynamics (MHD). For more details in this regard, see [12].

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